Exercise on Expectation & Variance - Worked Solutions.

1.

x	-2	-1	0	1	2
$P(X=x)$	0.15	0.25	0.30	0.20	0.10
$xP(X=x)$	-0.3	-0.25	0	0.2	0.2
$x^2P(X=x)$	0.6	0.25	0	0.2	0.4

a) $E(X) = \Sigma x P(X=x)$

$= -0.3 + (-0.25) + 0 + 0.2 + 0.2$

$= -0.55 + 0.4$

$= \underline{\underline{-0.15}}$

$Var(X) = E(X^2) - E^2(X)$ now $E(X^2) = \Sigma x^2 P(X=x)$

$= 0.6 + 0.25 + 0 + 0.2 + 0.4$

$= 1.45.$

so $Var(X) = 1.45 - (-0.15)^2$

$= \underline{\underline{1.4275}}$

b) i) $E(3X+5)$

$= 3E(X) + 5$

$= 3 \times (-0.15) + 5$

$= -0.45 + 5$

$= \underline{\underline{4.55}}.$

ii) $Var(X-2)$

$= Var(X)$

$= \underline{\underline{1.4275}}.$

iii) $Var(2X)$

$= 2^2 \, Var(X)$

$= 4 \times 1.4275$

$= \underline{\underline{5.71}}.$

2.

$E(T) = 2.5 \qquad E(W) = 3.6$

$Var(T) = 0.7 \qquad Var(W) = 1.2$

i) $E(3T - W)$

$= E(3T) - E(W)$

$= 3E(T) - E(W)$

$= 3 \times 2.5 - 3.6$

$= \underline{\underline{3.9}}$

ii) $E(T + \frac{1}{2}W)$

$= E(T) + E(\frac{1}{2}W)$

$= E(T) + \frac{1}{2}E(W)$

$= 2.5 + \frac{1}{2} \times 3.6$

$= \underline{\underline{4.3}}$.

iii) $Var(T - W)$

$= Var(T) + Var(W)$

$= 0.7 + 1.2$

$= \underline{\underline{1.9}}$.

iv) $Var(4T + 2W)$

$= Var(4T) + Var(2W)$

$= 4^2 Var(T) + 2^2 Var(W)$

$= 16 \times 0.7 + 4 \times 1.2$

$= \underline{\underline{16}}$.

3. fair D6 thrown. X = score on uppermost face

x	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

a) $E(X) = 3.5$, by inspection

$$E(X^2) = \sum x^2 P(X=x)$$
$$= \frac{1}{6}(1^2+2^2+3^2+4^2+5^2+6^2)$$
$$= \frac{1}{6} \times 91$$
$$= \frac{91}{6}$$

$$Var(X) = E(X^2) - E^2(X)$$
$$= \frac{91}{6} - \left(\frac{7}{2}\right)^2$$
$$= \frac{91}{6} - \frac{49}{4}$$
$$= \frac{35}{12}$$

$\therefore$ $\underline{\underline{E(X) = \frac{7}{2}, \ Var(X) = \frac{35}{12}.}}$

b) let X_i = score on dice i $\quad E(X_i) = \frac{7}{2}$ $\quad Var(X_i) = \frac{35}{12}$

let $T = X_1 + X_2 + X_3 + X_4$

$$E(T) = E(X_1+X_2+X_3+X_4)$$
$$= E(X_1)+E(X_2)+E(X_3)+E(X_4)$$
$$= \frac{7}{2} \times 4$$
$$\underline{\underline{= 14}}$$

$$Var(T) = Var(X_1+X_2+X_3+X_4)$$
$$= Var(X_1)+Var(X_2)+Var(X_3)+Var(X_4)$$
$$= 4 \times \frac{35}{12}$$
$$\underline{\underline{= \frac{35}{3}}}.$$

4. X = no. cars sold per week

x	0	1	2	3	4	5
$P(X=x)$	0.09	0.38	0.26	0.15	0.10	0.02

$$E(X) = \Sigma x P(X=x)$$
$$= 1.85$$

$$E(X^2) = \Sigma x^2 P(X=x)$$
$$= 4.87$$

$$Var(X) = E(X^2) - E^2(X)$$
$$= 4.87 - 1.85^2$$
$$= 1.4475$$

b) let commission, $C = 30X$

so $E(C) = E(30X)$
$= 30E(X)$
$= 30 \times 1.85$
$= 55.5$

$Var(C) = Var(30X)$
$= 30^2 Var(X)$
$= 900 \times 1.4475$
$= 1302.75$

Hence his expected commission is £55.50, with variance £^{2} 1302.75
(st. dev £36.09)

5.

let X = pay per week

$E(X) = 22$

$Var(X) = 3^2$

a) let N = new pay per week, at 8% rise

$N = 1{\cdot}08X$

so $E(N) = E(1{\cdot}08X)$
$= 1{\cdot}08\,E(X)$
$= 1{\cdot}08 \times 22$
$= 23{\cdot}76$

$Var(N) = Var(1{\cdot}08X)$
$= 1{\cdot}08^2\,Var(X)$
$= 1{\cdot}08^2 \times 3^2$
$= 3{\cdot}24^2$

so expected new pay of £23·76 with st. deviation £3·24.

b) let M = new pay, at £5 extra

$M = X + 5$

$E(M) = E(X+5)$
$= E(X) + 5$
$= 22 + 5$
$= 27$

$Var(M) = Var(X+5)$
$= Var(X)$
$= 3^2$

so her expected pay would be £27, and standard deviation left unchanged at £3.