ACCIALS Ex6F Solutions.

1. $X \sim U(3,6)$

a) $f(x) = \begin{cases} \frac{1}{3} & 3 \le x \le 6 \\ 0 & \text{otherwise} \end{cases}$

b) if $X \sim U(a,b)$ then $E(X) = \frac{a+b}{2}$

$Var(X) = \frac{(b-a)^2}{12}$

here $a=3, b=6$.

so $E(X) = \frac{3+6}{2} = \underline{\underline{\frac{9}{2}}}$

$Var(X) = \frac{(6-3)^2}{12} = \frac{9}{12} = \underline{\underline{\frac{3}{4}}}$

d) $P(X>5) = \int_5^6 \frac{1}{3} \, dx$

$= \underline{\underline{\frac{1}{3}}}.$

2. $X \sim U(-5, -2)$

$$f(x) = \begin{cases} k & -5 \le x \le -2 \\ 0 & \text{otherwise} \end{cases}$$

So $k = \frac{1}{3}$ as $-2-(-5)$
$= -2+5$
$= 3.$

b) $P(-4.3 < X < -2.8)$

$= \frac{1}{3}(-2.8-(-4.3))$

$= \frac{1}{3}(4.3-2.8)$

$= \frac{1}{3} \times 1.5$

$= \underline{\underline{0.5}}.$

c) $E(X) = \frac{-5-2}{2} = \underline{\underline{-\frac{7}{2}}}$

d) $Var(X) = \frac{(-2+5)^2}{12} = \frac{3^2}{12} = \underline{\underline{\frac{3}{4}}}$

3. a) $k = 1 + 4$

$\underline{\underline{k = 5}}$

so $X \sim U(1,5)$

b) $P(2{\cdot}1 < X < 3{\cdot}4) = \frac{1}{4}\ (3{\cdot}4 - 2{\cdot}1)$

$= \frac{1}{4}\,(1{\cdot}3)$

$= \underline{\underline{\frac{13}{40}}}$

c) $E(X) = \frac{1+5}{2} = \underline{\underline{3}}.$

d) $Var(X) = \frac{(5-1)^2}{12} = \frac{4^2}{12} = \underline{\underline{\frac{4}{3}}}.$

4 $f(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$ where $b>a$

$$E(X) = \int_a^b x f(x)\,dx$$

$$= \int_a^b x \cdot \frac{1}{b-a} \cdot dx$$

$$= \frac{1}{b-a}\left[\tfrac{1}{2}x^2\right]_a^b$$

$$= \frac{1}{2} \cdot \frac{1}{b-a} \cdot (b^2-a^2)$$

$$= \underline{\underline{\frac{b+a}{2}}}$$

$$E(X^2) = \int_a^b x^2 f(x)\,dx$$

$$= \frac{1}{b-a}\left[\tfrac{1}{3}x^3\right]_a^b$$

$$= \frac{1}{3} \cdot \frac{1}{b-a} \cdot b^3 - a^3$$

$$= \frac{b^3-a^3}{3(b-a)}$$

So $Var(X) = E(X^2) - E^2(X)$

$$= \frac{b^3-a^3}{3(b-a)} - \left[\frac{a+b}{2}\right]^2$$

$$= \frac{b^3-a^3}{3(b-a)} - \frac{(a+b)^2}{4}.$$

$$= \frac{1}{12} \cdot \frac{1}{b-a}\left(4(b^3-a^3) - 3(b-a)(a+b)^2\right)$$

$$= \frac{1}{12} \cdot \frac{1}{b-a}\left(4\,(b-a)(b^2+ab+a^2) - 3(b-a)\,(a+b)^2\right)$$

$$= \frac{1}{12}\left(4\,(b^2+ab+a^2) - 3\,(a^2+2ab+b^2)\right)$$

$$= \frac{1}{12}\left(4b^2+4ab+4a^2-3a^2-6ab-3b^2\right)$$

$$= \frac{1}{12}\left(b^2-2ab+a^2\right)$$

$$= \underline{\underline{\frac{(b-a)^2}{12}.}}$$

$\frac{a+b}{2} = 1 \Rightarrow a+b = 2$ ①

$\frac{(b-a)^2}{12} = \frac{4}{3} \Rightarrow (b-a)^2 = 16$

$b-a = \pm 4$ ②

$\underline{b+a = 2}$ ①

$2b = 2 \pm 4$

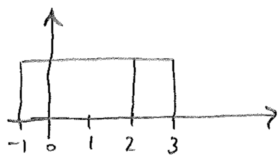
$b = 1 \pm 2$

$b = -1, 3$

$\Rightarrow a = 3, -1$

so $a = -1$, $b = 3$ as $b > a$

so $P(X<0) = \frac{1}{4}(0-(-1)) = \underline{\underline{\frac{1}{4}}}$



$P(X>x) = \frac{1}{4}$

$\Rightarrow x = 2$

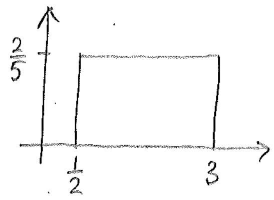
now $z + \sqrt{\frac{4}{3}} = 2$

$z = 2 - \frac{2}{\sqrt{3}}$

$z = 2 - \frac{2\sqrt{3}}{3}$

$z = \underline{\underline{\frac{6-2\sqrt{3}}{3}}}$

5.



so $X \sim U(\frac{1}{2}, 3)$

$$P(X<1.5) = \frac{2}{5}\left(1.5-\frac{1}{2}\right)$$
$$= \frac{2}{5} \times 1$$
$$= \frac{2}{5}$$

$$E(X) = \frac{\frac{1}{2}+3}{2}$$
$$= \frac{7/2}{2}$$
$$= \frac{7}{4}$$

$$P(X>E(X)) = \frac{2}{5}\left(3-\frac{7}{4}\right)$$
$$= \frac{2}{5}\cdot\frac{5}{4}$$
$$= \frac{1}{2}.$$

so $P(X_i < 1.5 \text{ and } X_j > \mu) = P(X_1 < 1.5 \cap X_2 > \mu) + P(X_1 > \mu \cap X_2 < 1.5)$

$= P(X_1 < 1.5)P(X_2 > \mu) + P(X_1 > \mu)P(X_2 < 1.5)$ as X_1 & X_2 are independent

$$= \frac{2}{5} \times \frac{1}{2} + \frac{1}{2} \times \frac{2}{5}$$
$$= \frac{1}{5} + \frac{1}{5}$$
$$= \frac{2}{5}.$$

6. $Y \sim U(32, 37)$

$$f(y) = \begin{cases} \frac{1}{5} & 32 \leq y \leq 37 \\ 0 & \text{otherwise} \end{cases}$$

$$E(Y) = \frac{32+37}{2} = 34\tfrac{1}{2}$$

$$Var(Y) = \frac{(37-32)^2}{12}$$

$$= \frac{5^2}{12}$$

$$= \frac{25}{12}.$$

$$P\left(34\tfrac{1}{2} - \sqrt{\tfrac{25}{12}} < Y < 34\tfrac{1}{2} + \sqrt{\tfrac{25}{12}}\right)$$

$$= \frac{1}{5}\left[\left(34\tfrac{1}{2} + \sqrt{\tfrac{25}{12}}\right) - \left(34\tfrac{1}{2} - \sqrt{\tfrac{25}{12}}\right)\right]$$

$$= \frac{1}{5}\left[\cdot 2 \cdot \frac{5}{\sqrt{12}}\right]$$

$$= \frac{2}{\sqrt{12}}$$

$$= \frac{2}{2\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}}$$

$$= \frac{\sqrt{3}}{3}.$$